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COMMENT

On spontaneous pair creation

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Abstract. It is shown that the gap separating the positive and negative energy states of the Dirac electron in the inhomogeneous, time-independent magnetic field does not shrink when the field gets stronger.

Recent interest in superconducting cosmic strings (SCS) (Witten 1985) has brought up the question of the stability of QED vacuum in the presence of a very strong inhomogeneous magnetic field.

The SCS arise in some specific spontaneously broken gauge theories. They have a thickness of 10^{-31} cm and are of infinite length or form loops of arbitrary size. They can behave as superconducting wires carrying electric currents (e.g., 10^{20} A) and thus produce a strong inhomogeneous magnetic field in their vicinity.

It was claimed that the spontaneous creation of e^+e^- may take place in such a field due to the closing of the energy gap which separates the positive and negative energy states of the Dirac electron. For the free electron the width of the gap is equal to $2mc^2$. On the other hand, in the presence of the electric Coulomb potential this gap shrinks and in the case of critical field (corresponding to the point-like charge equal to $137e$) it closes (Greiner *et al* 1985). This mechanism gives rise to spontaneous creation of a e^+e^- pair. However, in the presence of an arbitrary strong homogeneous and static magnetic field this does not occur (Itzykson and Zuber 1980).

The following heuristic argument shows that the energy gap width does not shrink even if the magnetic field is inhomogeneous. Consider the Hamiltonian ($\hbar = c = 1$)

$$\hat{H} = \hat{\boldsymbol{\alpha}}(\hat{\boldsymbol{p}} - e\boldsymbol{A}(\boldsymbol{r})) + \hat{\beta}m \tag{1}$$

which describes the Dirac electron in the static but otherwise arbitrary magnetic field $\boldsymbol{H}(\boldsymbol{r}) = \mathrm{rot}\,\boldsymbol{A}(\boldsymbol{r})$.

The eigenenergy problem is defined by the equation

$$\hat{H}\psi_{\mathrm{E}}(\boldsymbol{r}) = E\psi_{\mathrm{E}}(\boldsymbol{r}) \tag{2}$$

or explicitly

$$[\hat{\boldsymbol{\alpha}}(\hat{\boldsymbol{p}} - e\boldsymbol{A}(\boldsymbol{r})) + \hat{\beta}m]\psi_{\mathrm{E}}(\boldsymbol{r}) = E\psi_{\mathrm{E}}(\boldsymbol{r}). \tag{2a}$$

The states ψ_{E} also satisfy

$$\hat{H}^2\psi_{\mathrm{E}}(\boldsymbol{r}) = E^2\psi_{\mathrm{E}}(\boldsymbol{r}) \tag{3}$$

with the squared Hamiltonian equal to

$$\hat{H}^2 = [\hat{\boldsymbol{\alpha}}(\hat{\boldsymbol{p}} - e\boldsymbol{A}(\boldsymbol{r}))]^2 + m^2. \tag{4}$$

The mixed terms proportional to m vanish because the matrices $\hat{\beta}$ and $\hat{\boldsymbol{\alpha}}$ anticommute. Now (3) may be rewritten

$$[\hat{\boldsymbol{\alpha}}(\hat{\boldsymbol{p}} - e\boldsymbol{A}(\boldsymbol{r}))]^2 \psi_{\mathrm{E}}(\boldsymbol{r}) = (E^2 - m^2)\psi_{\mathrm{E}}(\boldsymbol{r}). \tag{5}$$

As the operator $[\hat{\boldsymbol{\alpha}}(\hat{\boldsymbol{p}} - e\boldsymbol{A}(\boldsymbol{r}))]^2$ is a square of the self-adjoint operator $\hat{\boldsymbol{\alpha}}(\hat{\boldsymbol{p}} - e\boldsymbol{A}(\boldsymbol{r}))$ it has only non-negative eigenvalues. Thus for every ψ_{E}

$$E^2 - m^2 \geqslant 0 \tag{6}$$

or equivalently

$$|E| \geqslant m \tag{6a}$$

which completes the proof.

The Dirac electron in an inhomogeneous magnetic field was considered by Stanciu (1967) and then by Achuthan *et al* (1979). The latter authors suggest that for the field configuration $\boldsymbol{H} = H\,\mathrm{sech}^2(ay)\boldsymbol{e}_z$ the energy gap shrinks and for certain values of the parameters it closes, indicating the effect of spontaneous pair creation, thus contradicting the above general argument.

If it were true, it would suggest the logical possibility of a similar effect in the magnetic field of the SCS (as initially claimed by Amsterdamski and O'Connor (1987)). The aim of this comment is to note explicitly that the analysis of Achuthan *et al* (1979) is incomplete. The energy gap does not shrink to zero. Indeed, as is shown in the appendix, $|E| \geqslant mc^2$.

Our numerical study of the Dirac electron in the field of a superconducting cosmic string leads to the same conclusion: $|E| \geqslant mc^2$, excluding the possibility of spontaneous e^+e^- creation (at least in this way).

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Appendix†

The Dirac equation for an inhomogeneous magnetic field

$$\boldsymbol{H}(y) = H\,\mathrm{sech}^2(ay)\boldsymbol{e}_z \qquad a>0 \qquad H>0 \tag{A1}$$

is ($\hbar = c = m = 1$)

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \frac{e^2H^2}{a^2} + \frac{2p_x eH}{a}\tanh(ay) - eH\left(\frac{eH}{a^2} - s\right)\mathrm{sech}^2(ay)\right]\psi_s\chi_s$$
$$= (E_s^2 - 1 - p_x^2 - p_z^2)\psi_s\chi_s. \tag{A2}$$

† We use the same conventions as Stanciu (1967) and Achuthan *et al* (1979). More details can be found in those works.

The solutions of this equation were found by Stanciu (1967) and the eigenenergies are given by

$$E_s^2 = 1 + p_x^2 + p_z^2 + 2NeH - a^2N^2 - \frac{e^2H^2p_x^2}{(eH - a^2N)^2} \tag{A3}$$

where

$$N = n + \tfrac{1}{2} + \tfrac{1}{2}s \tag{A4a}$$

for electrons and

$$N = n + \tfrac{1}{2} - \tfrac{1}{2}s \tag{A4b}$$

for positrons, and the quantum number n must satisfy the relationship (Stanciu 1967)

$$0 \leqslant n \leqslant \xi/a - |\xi p_x/a^2|^{1/2} - \tfrac{1}{2}s - \tfrac{1}{2} \tag{A5a}$$

for electrons and

$$0 \leqslant n \leqslant \xi/a - |\xi p_x/a^2|^{1/2} + \tfrac{1}{2}s - \tfrac{1}{2} \tag{A5b}$$

for positrons, or equivalently ($s = \pm 1$)

$$\tfrac{1}{2} \pm \tfrac{1}{2}s \leqslant N \leqslant \xi/a - |\xi p_x/a^2|^{1/2} \tag{A5c}$$

where $\xi = eH/a$. If (A5) is not satisfied, there are no bound states† for such n.

First we show that this solution has the required property $E_s^2 \geqslant 1$. We define the function $f(p_x, \xi, Na)$ such that

$$E_s^2 = 1 + p_z^2 + f(p_x, \xi, Na) \tag{A6}$$

$$f(p_x, \xi, Na) = p_x^2 + \xi^2 - (\xi - Na)^2 - \frac{\xi^2 p_x^2}{(\xi - Na)^2}. \tag{A7}$$

$f(p_x, \xi, Na)$ may be rewritten

$$f(p_x, \xi, Na) = \frac{Na(Na - 2\xi)[p_x^2 - (\xi - Na)^2]}{(\xi - Na)^2}. \tag{A8}$$

Of course, $Na \geqslant 0$. From condition (A5c) we obtain $Na - \xi \leqslant -|\xi p_x|^{1/2} \leqslant 0$ and we can establish the sign of the second term in (A8):

$$Na - 2\xi \leqslant Na - \xi \leqslant -|\xi p_x|^{1/2} \leqslant 0. \tag{A9}$$

From (A5c) we can also conclude $\xi|p_x| \leqslant (\xi - Na)^2$ and since $0 \leqslant \xi - |\xi p_x|^{1/2}$ then $|p_x| \leqslant \xi$ and the last term in (A8) has the correct sign:

$$p_x^2 - (\xi - Na)^2 \leqslant \xi|p_x| - (\xi - Na)^2 \leqslant 0. \tag{A10}$$

Thus we obtain $f(p_x, \xi, Na) \geqslant 0$ and due to (A6) this results in

$$E_s^2 \geqslant 1 + p_z^2 \geqslant 1. \tag{A11}$$

Achuthan *et al* (1979) suggest that for scaling

$$p_x \simeq k > 0 \qquad eH - a^2 \simeq k \tag{A12}$$

there is a solution $E_s^2 = 0$ for $N = 1$.

† We do not discuss the scattering states as they were not the source of confusion.

If we express the bound state condition (A5c) using scaling (A12) we obtain (with $N=1$)

$$aeHk \leqslant (eH-a^2)^2 = k^2 \tag{A13}$$

or, substituting a from (A12),

$$k^2 + k(eH)^2 - (eH)^3 \geqslant 0. \tag{A14}$$

Note that not all combinations of k and eH satisfy this inequality.

If we put $E_s^2 = 0$ then equation (A3) for scaling (A12) has a formal solution

$$eH = \tfrac{1}{2}\{1 + [5 + 4(k + k^2 + p_z^2)]^{1/2}\}. \tag{A15}$$

Now the question is whether solution (A15) is consistent with condition (A14). Substituting (A15) into (A14) one can get the answer:

$$2k^3 + k^2 + p_z^2(2k-3) - 4 \geqslant (2 + k^2 + p_z^2)[1 + 4(1 + k + k^2 + p_z^2)]^{1/2} \tag{A16}$$

or

$$0 \geqslant 1 + 4k + 11k^2 + 8k^3 + 5k^4 + p_z^2(3 + 8k + 12k^2 + 4k^3 + k^4) + p_z^4(3 + 4k + 2k^2) + p_z^6. \tag{A17}$$

The RHS of (A17) is evidently greater than zero. Solution (A15) has no physical meaning.

References

Achuthan P, Chandramohan T and Venkatesan K 1979 *J. Phys. A: Math. Gen.* **12** 2521

Amsterdamski P and O'Connor D J 1987 private communication

Greiner W, Müller B and Rafelski J 1985 *Quantum Electrodynamics of Strong Fields* (Berlin: Springer)

Itzykson C and Zuber J-B 1980 *Quantum Field Theory* (New York: McGraw-Hill)

Stanciu G N 1967 *J. Math. Phys.* **8** 2043

Witten E 1985 *Nucl. Phys.* B **249** 557